MA 3046 - Matrix Analysis

Problem Set 4 - $\mathbf{Q}\,\mathbf{R}$ Factorization and Least Squares

1. Consider the subspaces:

$$\mathrm{U} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathrm{V} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

a. Determine whether U and V are complementary subspaces.

b. Determine whether U and V are complementary orthogonal subspaces.

c. Find the matrix for the projector onto U along V , using the standard representation:

$$\mathbf{P} = \mathbf{B} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{B}^{-1}.$$

where $\mathbf{B}$ is the matrix whose columns represent, sequentially, bases for U and V .

d. If U and V are orthogonal complements, find the matrix for the projection onto U using the representation:

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^H \mathbf{A}\right)^{-1} \mathbf{A}^H$$

where $\mathbf{A}$ is the matrix whose columns are a basis for U and compare that the result from part c. above.

e. For the matrix $\mathbf{P}$ found in part c., show by direct computation that $\mathbf{P}^2 = \mathbf{P}$.

f. For the vector $\mathbf{x} = [\ 2\ \ 3\ \ 1\ ]^T$, find the projector of $\mathbf{x}$ onto U along V :

(1.) Using $\mathbf{P}$ as determined in part c. above.

(2.) By finding the coordinates of $\mathbf{x}$ in terms of the basis for U and V , and compare the two.

g. Compare the projector of $\mathbf{x}$ onto U along V computed in part f. above with the original vector $\mathbf{x}$ and explain any differences.

h. Repeat parts f. and g. for the vector $\mathbf{y} = [\ 2\ \ -1\ \ \ 1]^T$.

2. Repeat problem 2 for the subspaces

$$\mathrm{U} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathrm{V} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and the vectors $\mathbf{x} = [\,0 \;\; 1 \;\; 4\,]^T$ and $\mathbf{y} = [\,2 \;\; -1 \;\; 1\,]^T$.

3. Find the matrix $\mathbf{P}$ which projects an arbitrary vector in $\Re^5$ onto the subspace spanned by:

$$\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{a}^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Show directly that $\text{Col}(\mathbf{P})$ is identical to the span of $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$. Also show directly that $\mathbf{P}^2 = \mathbf{P}$.

4. Use the Gram-Schmidt method to produce an **orthonormal** basis for the column space of

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$

5. Use the classic Gram-Schmidt method to produce a $\mathbf{Q}\,\mathbf{R}$ factorization of

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix}$$

6. Use the modified Gram-Schmidt method to produce a $\mathbf{Q}\,\mathbf{R}$ factorization of

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix}$$

7. Find a sequence of upper triangular matrices $\tilde{\mathbf{R}}^{(i)}$, each corresponding to a single step of the classic Gram-Schmidt method, such that the matrix $\mathbf{Q}$ in the $\mathbf{QR}$ factorization in problem 5 can be written:

$$\mathbf{Q} = \mathbf{A}\tilde{\mathbf{R}}^{(1)}\tilde{\mathbf{R}}^{(2)}\tilde{\mathbf{R}}^{(3)}$$

Show also by direct computation that the matrix $\mathbf{R}$ from the $\mathbf{QR}$ factorization can be written in terms of the inverses of these $\tilde{\mathbf{R}}^{(i)}$ as:

$$\mathbf{R} = \left(\tilde{\mathbf{R}}^{(3)}\right)^{-1}\left(\tilde{\mathbf{R}}^{(2)}\right)^{-1}\left(\tilde{\mathbf{R}}^{(1)}\right)^{-1}$$

8. Find a sequence of upper triangular matrices $\mathbf{R}^{(i)}$, each corresponding to a single step of the modified Gram-Schmidt method, such that the matrix $\mathbf{Q}$ in the $\mathbf{QR}$ factorization in problem 6 can be written:

$$\mathbf{Q} = \mathbf{A}\mathbf{R}^{(1)}\mathbf{R}^{(2)}\mathbf{R}^{(3)}$$

Show also by direct computation that the matrix $\mathbf{R}$ from the $\mathbf{QR}$ factorization can be written in terms of the inverses of these $\mathbf{R}^{(i)}$ as:

$$\mathbf{R} = \left(\mathbf{R}^{(3)}\right)^{-1}\left(\mathbf{R}^{(2)}\right)^{-1}\left(\mathbf{R}^{(1)}\right)^{-1}$$

9. Use the modified Gram-Schmidt method to produce a reduced $\mathbf{Q}\,\mathbf{R}$ factorization of

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 6 \\ -1 & -4 \\ 1 & 4 \end{bmatrix}$$

10. Create an orthogonal plane (Givens) rotation matrix ($\mathbf{Q}$) which uses the third row of:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & -3 & -1 \\ 1 & -1 & 3 \\ 1 & -3 & -3 \end{bmatrix}$$

to zero out the element currently in the (1,3) position.

11. Produce a reflection (Householder) matrix ($\mathbf{Q}$) and the associated vector ($\mathbf{u}$) which will zero out the elements below the *second row* in the first column of:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ -2 & -3 & -1 \\ 4 & -1 & 3 \\ 2 & -3 & -3 \\ 5 & 1 & -1 \end{bmatrix}$$

12. Consider the full **QR** factorization of the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & \frac{3}{6} & -\frac{3}{6} & -\frac{3}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the least squares solution to

$$\mathbf{Ax} = \begin{bmatrix} -43 \\ -3 \\ -10 \\ -16 \end{bmatrix}$$

by using both the normal equations and the **QR** factorization shown. Also confirm that your residual is orthogonal to the column space of **A**